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HYPERSTABILITY OF A SUM FORM FUNCTIONAL EQUATION RELATED DISTANCE MEASURES

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Abstract. The functional equation related to a distance measure

$$f(pr, qs) + f(ps, qr) = M(r, s)f(p, q) + M(p, q)f(r, s)$$

can be generalized a sum form functional equation as follows

$$\frac{1}{n}\sum_{i=0}^{n-1} f(P \cdot \sigma_i(Q)) = M(Q)f(P) + M(P)f(Q)$$

where f, g is information measures, P and Q are the set of n-array discrete measure, and σ_i is a permutation for each $i = 0, 1, \cdots, n-1$. In this paper, we obtain the hyperstability of the above type functional equation.

1. Introduction

Throughout this paper, let $(G, \cdot)$ denote a group and X a real normed algebra. Also let $\mathbb{R}$ and $\mathbb{C}$ denote the set of real and complex numbers, respectively. Let $\mathbb{R}_+ = \{x \in \mathbb{R} \,|\, x > 0\}$ be a set of positive real numbers.

Chung et al. [2] solved the following functional equation related to distance measures

$$f(pr, qs) + f(ps, qr) = (r + s)f(p, q) + (p + q)f(r, s) \tag{1.1}$$

and

$$f(pr, qs) + f(ps, qr) = f(p, q)f(r, s) \tag{1.2}$$

for all p, q, r, s in the open interval $(0, 1)$ and Elfen et al. [1] solved the general solution $f : G \times G \longrightarrow \mathbb{C}$ of the following functional equation

$$f(pr, qs) + f(sp, rq) = 2f(p, q) + 2f(r, s)$$

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for all $p, q, r, s \in G$.

Y. W Lee and G. H. Kim [3] investigate the superstability for a generalized form of the equation (1.2)

$$\frac{1}{n}\sum_{i=0}^{n-1} f(P\cdot\sigma_i(Q)) = f(P)f(Q)$$

for all $P, Q \in G^n$, where $f : G^n \to G$ is a function and σ_i is a permutation on G.

In 2001, G. Maksa and Z. Páles [4] proved a new type of stability of a class of linear functional equation

$$f(s) + f(t) = \frac{1}{n}\sum_{i=1}^{n} f(s\varphi_i(t)), \qquad s, t \in S \tag{1.3}$$

where f is a functional on a semigroup $S := (S, \cdot)$ and where $\varphi_1, \cdots, \varphi_n : S \to S$ pairwise distinct automorphisms of S such that the set $\{\varphi_1, \cdots, \varphi_n\}$ is a group with the operation of composition of mappings. More precisely, they proved that if the error bound for the difference of two sides of (1.3) satisfies a certain asymptotic property, then in fact, the two sides have to be equal. Such a phenomenon is called the *hyperstability* of the functional equation on S.

In 2015, M. Sirouni and S. Kabbaj [6] investigated of the hyperstability of an Euler-Lagrange type quadratic functional equation

$$f(x+y) + \frac{f(x-y) + f(y-x)}{2} = 2f(x) + 2f(y)$$

in class of functions from an abelian group into Banach space. This equation is established the general solution and stability by M. J. Rassias [5].

In this paper, we obtain a generalized form of the equation (1.1) related to distance measures and prove the hyperstability of following functional equations

$$\frac{1}{n}\sum_{i=0}^{n-1} f(P\cdot\sigma_i(Q)) = M(Q)f(P) + M(P)f(Q), \tag{1.4}$$

$$\begin{aligned}\frac{1}{n}\sum_{i=0}^{n-1} f(P\cdot\sigma_i(Q)) =& M(Q)f(P) + M(P)f(Q)\\ &+ M(P\cdot Q)\theta(M(P), M(Q)),\end{aligned} \tag{1.5}$$

$$\frac{1}{n}\sum_{i=0}^{n-1} f(P\cdot\sigma_i(Q)) = f(P)+f(Q) \tag{1.6}$$

and

$$\frac{1}{n}\sum_{i=0}^{n-1} f(P\cdot\sigma_i(Q)) = f(P)+f(Q)+\theta(M(P),M(Q)). \tag{1.7}$$

2. Hyperstability of equations

For all $P=(p_1,p_2,\ldots,p_n), Q=(q_1,q_2,\ldots,q_n)\in G^n$, let $\sigma_i : G^n\to G^n$ be a permutation given by

$$\sigma_0(p_1,p_2,\ldots,p_n) = (p_1,p_2,\ldots,p_n),$$
$$\sigma_i(p_1,p_2,\ldots,p_n) = (p_{i+1},\ldots p_n,p_1,p_2,\ldots,p_i)$$

for each $i=1,\ldots,n-1$ and $P\cdot Q=(p_1q_1,p_2q_2,\ldots,p_nq_n)$.
It can be easily checked that for all $P,Q,W\in G$ and $i,j=0,1,\cdots,n-1$

(a) $\sigma_n(P)=\sigma_0 P$,

(b) $\sigma_{n+j}(P)=\sigma_j P$,

(c) $(P\cdot\sigma_i(Q))\cdot\sigma_j(W)=P\cdot(\sigma_i(Q)\cdot\sigma_j(W))$,

(d) $P\cdot\sigma_i(Q\cdot\sigma_j(W))=P\cdot\sigma_i(Q)\cdot(\sigma_i\sigma_j(W))$,

(e) $\displaystyle\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\sigma_i(Q)\cdot\sigma_j(W)=\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\sigma_i(Q)\cdot\sigma_{i+j}(W)$.

For convenience, any solution $M: G^n\to X$ of the functional equation

$$M(P\cdot Q)=M(P)M(Q),$$
$$M(\sigma_i(P))=M(P),\quad (P,Q\in G^n, i=0,1,\cdots,n-1)$$

is called a *strong multiplicative* function. For example, consider a function $M:\mathbb{R}^n\to\mathbb{R}$ defined by $M(P)=p_1p_2\cdots p_n=\prod_{i=1}^n p_i$ for any $P=(p_1,p_2,\cdots,p_n)\in\mathbb{R}^n$. Then M is strong multiplicative.

Note that any solution $F: G\times G\to X$ of the functional equation

$$F(x,y)+F(xy,z)=F(x,yz)+F(y,z),\qquad (x,y,z\in G)$$

is called a *cocycle* on $G\times G$ into X and the equation is called the *cocycle equation.* It is well known that the cocycle equation play an important role in the hyperstability.

Lemma 2.1. *Let $f : G^n \to X$ be an arbitrary function, $\theta : X \times X \to X$ a cocycle and $M : G^n \to X$ a strong multiplicative function. Then the function $F : G^n \to S$ defined by*

$$F(P,Q) = f(P)+f(Q)-\frac{1}{n}\sum_{i=0}^{n-1} f(P\cdot\sigma_i(Q))+\theta(M(P),M(Q)), \quad (P,Q \in G^n)$$

satisfies the following functional equation

$$F(P,Q) + \frac{1}{n}\sum_{i=0}^{n-1} F(P\cdot\sigma_i(Q),W) = F(Q,W) + \frac{1}{n}\sum_{i=0}^{n-1} F(P, Q\cdot\sigma_i(W)) \tag{2.1}$$

for all $P, Q, W \in G^n$.

Proof. (1) For all $P, Q, W \in G^n$, we have

$$\begin{aligned}
&F(P,Q) + \frac{1}{n}\sum_{i=0}^{n-1} F(P\cdot\sigma_i(Q),W) \\
=&f(P) + f(Q) - \frac{1}{n}\sum_{i=0}^{n-1} f(P\cdot\sigma_i(Q)) + \theta(M(P),M(Q)) \\
&+ \frac{1}{n}\sum_{i=0}^{n-1}\Big\{f(P\cdot\sigma_i(Q)) + f(W) - \frac{1}{n}\sum_{j=0}^{n-1} f(P\cdot\sigma_i(Q)\cdot\sigma_j(W))\Big\}. \\
&+ \frac{1}{n}\sum_{i=0}^{n-1}\theta(M(P\cdot\sigma_i(Q)),M(W)) \\
=&f(P) + f(Q) + f(W) - \frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} f(P\cdot\sigma_i(Q)\cdot\sigma_j(W)) \\
&+ \theta(M(P),M(Q)) + \theta(M(P)M(Q),M(W)).
\end{aligned}$$

(2) Similarly, for all $P, Q, W \in G^n$, we deduce

$$\begin{aligned}
&F(Q,W) + \frac{1}{n}\sum_{i=0}^{n-1} F(P, Q\cdot\sigma_i(W)) \\
=&f(Q) + f(W) - \frac{1}{n}\sum_{i=0}^{n-1} f(Q\cdot\sigma_i(W)) + \theta(M(Q), M(W)) \\
&+ \frac{1}{n}\sum_{i=0}^{n-1}\Big\{f(P) + f(Q\cdot\sigma_i(W)) - \frac{1}{n}\sum_{j=0}^{n-1} f(P\cdot\sigma_j(Q\cdot\sigma_i(W)))\Big\} \\
&+ \frac{1}{n}\sum_{i=0}^{n-1}\theta(M(P), M(Q\cdot\sigma_i(W)) \\
=&f(Q) + f(W) + f(P) - \frac{1}{n^2}\sum_{i=0}^{n-1}\sum_{j=0}^{n-1} f(P\cdot\sigma_i((Q\cdot\sigma_i(W))) \\
&+ \theta(M(Q), M(W)) + \theta(M(P), M(Q)(W)).
\end{aligned}$$

Since, for any $P, Q, W \in G^n$,

$$\begin{aligned}
&\theta(M(P), M(Q)) + \theta(M(P)M(Q), M(W)) \\
&\quad = \theta(M(Q), M(W)) + \theta(M(P), M(Q)M(W)).
\end{aligned}$$

The functional equation (2.1) turn out to be valid. □

Theorem 2.2. *Let $\varepsilon > 0$ and $M : G^n \to \mathbb{R}_+$ be a strong multiplicative function with $M(P_0) > 1$ for some $P_0 \in G^n$. Assume also that a function $\theta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a cocycle and a function $f : G^n \to \mathbb{R}$ satisfy the inequality*

$$\begin{aligned}
&\Big|\frac{1}{n}\sum_{i=0}^{n-1} f(P\cdot\sigma_i(Q)) - M(Q)f(P) - M(P)f(Q) \\
&\quad - M(P\cdot Q)\theta(M(p), M(Q))\Big| \le \varepsilon
\end{aligned} \tag{2.2}$$

for all $P, Q \in G^n$. Then f is a solution of the functional equation (1.5). That is, for all $P, Q \in G^n$

$$\frac{1}{n}\sum_{i=0}^{n-1} f(P\cdot\sigma_i(Q)) = M(Q)f(P) + M(P)f(Q) + M(P\cdot Q)\theta(M(p), M(Q)).$$

Proof. For any $P, Q \in G^n$, we define

$$L(P) := \frac{f(P)}{M(P)}$$

and

$$F(P,Q) := L(P)+L(Q)-\frac{1}{n}\sum_{i=0}^{n-1} L(P\cdot\sigma_i(Q))+M(P\cdot Q)\theta(M(p),M(Q)).$$

With these notations, the stability inequality (2.2) reduces to

$$\begin{aligned}\Big|F(P,Q)\Big| &= \Big|\frac{1}{n}\sum_{i=0}^{n-1} L(P\cdot\sigma_i(Q)) - L(P) - L(Q) + \theta(M(p),M(Q))\Big|\\ &\leq \frac{\varepsilon}{M(P)M(Q)}\end{aligned}$$

for all $P, Q \in G^n$. By Lemma 2.1, we have

$$F(P,Q)+\frac{1}{n}\sum_{i=0}^{n-1} F(P\cdot\sigma_i(Q),W) = F(Q,W)+\frac{1}{n}\sum_{i=0}^{n-1} F(P,Q\cdot\sigma_i(W))$$

for all $P, Q, W \in G^n$. Letting $P = P_0^k$, we have, for any $Q, W \in G^n$

$$\begin{aligned}\Big|F(Q,W)\Big| &\leq \Big|\frac{1}{n}\sum_{i=0}^{n-1} F(P_0^k, Q\cdot\sigma_i(W))\Big|\\ &+\Big|F(P_0^k,Q)\Big| + \Big|\frac{1}{n}\sum_{i=0}^{n-1} F(P_0^k\cdot\sigma_i(Q),W)\Big|\\ &\leq \frac{1}{n}\sum_{i=0}^{n-1}\frac{\varepsilon}{M(P_0)^k M(Q\cdot W)} + \frac{\varepsilon}{M(P_0)^k M(Q)} + \frac{1}{n}\sum_{i=0}^{n-1}\frac{\varepsilon}{M(P_0)^k M(Q\cdot W)}\\ &\leq \frac{1}{M(P_0)^k}\Big(\frac{2n\varepsilon}{M(Q\cdot W)} + \frac{\varepsilon}{M(Q)}\Big)\\ &\longrightarrow 0\end{aligned}$$

as $k \to \infty$. And so $F(Q,W) = 0$ for all $Q, W \in G^n$. Thus f is a solution of (1.5). □

By Theorem 2.2 with $\theta = 0$, we have the following theorem.

THEOREM 2.3. *Let $\varepsilon > 0$ and $M : G^n \to \mathbb{R}_+$ be a strong multiplicative function with $M(P_0) > 0$ for some $P_0 \in G^n$ Assume also that a function $f : G^n \to \mathbb{R}$ satisfy the inequality*

$$\Big|\frac{1}{n}\sum_{i=0}^{n-1} f(P\cdot\sigma_i(Q)) - M(Q)f(P) - M(P)f(Q)\Big| \leq \varepsilon$$

for all $P, Q \in G^n$. Then f is a solution of the functional equation (1.4). That is, for all $P, Q \in G^n$

$$\frac{1}{n}\sum_{i=0}^{n-1} f(P \cdot \sigma_i(Q)) = M(Q)f(P) + M(P)f(Q).$$

COROLLARY 2.4. *Let $\varepsilon > 0$ and a function $f : \mathbb{R}_+^n \to \mathbb{R}$ satisfy the inequality*

$$\Big|\frac{1}{n}\sum_{i=0}^{n-1} f(P \cdot \sigma_i(Q)) - (\prod_{i=1}^{n} q_i)f(P) - (\prod_{i=1}^{n} p_i)f(Q)\Big| \le \varepsilon$$

for all $P = (p_1, \cdots, p_n), Q = (q_1, \cdots, q_n) \in R_+^n$. Then

$$\frac{1}{n}\sum_{i=0}^{n-1} f(P \cdot \sigma_i(Q)) = (\prod_{i=1}^{n} q_i)f(P) + (\prod_{i=1}^{n} p_i)f(Q)$$

for all $P = (p_1, \cdots, p_n), Q = (q_1, \cdots, q_n) \in R_+^n$.

Proof. Let $M(P) = \prod_{i=1}^{n} p_i = p_1 p_2 \cdots p_n$ for any $P = (p_1, p_2 \cdots p_n) \in R_+^n$. Then M is a strong multiplicative function and $M(P_0) = 2$ for $P_0 = (2, 1, \cdots, 1)$. By Theorem 2.3, we complete the proof. □

THEOREM 2.5. *Let $\varepsilon : G^n \times G^n \to \mathbb{R}$ be function for which there exists a sequence $(W_k)_{k\in\mathbb{N}}$ of elements of G^n satisfying the following condition:*

$$\lim_{k\to\infty} \varepsilon(W_k \cdot P, Q) = 0, \qquad P, Q \in G^n.$$

Also let $M : G^n \to X$ be a strong multiplicative function, a function $\theta : S \times S \to X$ be a cocycle, and a function $f : G^n \to X$ satisfy the inequality

$$\Big\|\frac{1}{n}\sum_{i=0}^{n-1} f(P \cdot \sigma_i(Q)) - f(P) - f(Q) + \theta(M(P), M(Q))\Big\| \le \varepsilon(P, Q)$$

for all $P, Q \in G^n$. Then f is a solution of the functional equation (1.7). That is, for all $P, Q \in G^n$

$$\frac{1}{n}\sum_{i=0}^{n-1} f(P \cdot \sigma_i(Q)) - f(P) - f(Q) + \theta(M(P), M(Q)).$$

Proof. For any $P, Q \in G^n$, we define

$$F(P, Q) := L(P) + L(Q) - \frac{1}{n}\sum_{i=0}^{n-1} L(P \cdot \sigma_i(Q)) + M(P \cdot Q)\theta(M(p), M(Q)).$$

By Lemma 2.1, we have

$$F(P,Q)+\frac{1}{n}\sum_{i=0}^{n-1}F(P\cdot\sigma_i(Q),W)=F(Q,W)+\frac{1}{n}\sum_{i=0}^{n-1}F(P,Q\cdot\sigma_i(W))$$

for all $P,Q,W\in G^n$. Letting $P=W_k\cdot P$, we have, for any $Q,W\in G^n$

$$\begin{aligned}\Big|F(Q,W)\Big| &\le \Big|\frac{1}{n}\sum_{i=0}^{n-1}F(W_k\cdot P,Q\cdot\sigma_i(W))\Big|\\ &+\Big|F(W_k\cdot P,Q)\Big|+\Big|\frac{1}{n}\sum_{i=0}^{n-1}F(W_k\cdot P\cdot\sigma_i(Q),W)\Big|\\ &\le \frac{1}{n}\sum_{i=0}^{n-1}\varepsilon(W_k\cdot P,Q\cdot\sigma_i(W))+\varepsilon(W_k\cdot P,Q)+\frac{1}{n}\sum_{i=0}^{n-1}\varepsilon(W_k\cdot P\cdot\sigma_i(Q),W)\\ &\longrightarrow 0\end{aligned}$$

as $k\to\infty$. And so $F(Q,W)=0$ for all $Q,W\in G^n$. Thus f is a solution of (1.7).

□

By Theorem 2.5 with $\theta=0$, we have the following theorem.

THEOREM 2.6. *Let $\varepsilon : G^n\times G^n\to\mathbb{R}$ be function for which there exists a sequence $(W_k)_{k\in\mathbb{N}}$ of elements of G^n satisfying the following condition:*

$$\lim_{k\to\infty}\varepsilon(W_k\cdot P,Q)=0,\qquad P,Q\in G^n.$$

Assume also that a function $f:G^n\to X$ satisfy the inequality

$$\Big\|\frac{1}{n}\sum_{i=0}^{n-1}f(P\cdot\sigma_i(Q))-f(P)-f(Q)\Big\|\le\varepsilon(P,Q)$$

for all $P,Q\in G^n$. Then f is a solution of the functional equation (1.6). That is, for all $P,Q\in G^n$

$$\frac{1}{n}\sum_{i=0}^{n-1}f(P\cdot\sigma_i(Q))=f(P)+f(Q).$$

COROLLARY 2.7. *Let $\varepsilon : G^n\times G^n\to\mathbb{R}_+$ be function for which there exists a sequence $(W_k)_{k\in\mathbb{N}}$ of elements of G^n satisfying the following condition:*

$$\lim_{k\to\infty}\varepsilon(W_k\cdot P,Q)=0,\qquad P,Q\in G^n.$$

and let a function $\theta : S \times S \to X$ be a cocycle. Assume also that a function $f : G^n \to X$ satisfy the inequality

$$\Big|\Big|\frac{1}{n}\sum_{i=0}^{n-1} f(P\cdot\sigma_i(Q)) - f(P) - f(Q) + \theta(\prod_{i=1}^{n} p_i, \prod_{i=1}^{n} q_i)\Big|\Big| \leq \varepsilon(P,Q)$$

for all $P = (p_1, \cdots, p_n), Q = (q_1, \cdots, q_n) \in G^n$. Then, for all $P = (p_1, \cdots, p_n)$, $Q = (q_1, \cdots, q_n) \in G^n$

$$\frac{1}{n}\sum_{i=0}^{n-1} f(P\cdot\sigma_i(Q)) = f(P) + f(Q) + \theta(\prod_{i=1}^{n} p_i, \prod_{i=1}^{n} q_i).$$

Corollary 2.8. *Assume that a function $f : \mathbb{R}_+^n \to X$ satisfy the inequality*

$$\Big|\Big|\frac{1}{n}\sum_{i=0}^{n-1} f(P\cdot\sigma_i(Q)) - f(P) - f(Q)\Big|\Big| \leq \prod_{i=1}^{n} p_i \prod_{i=1}^{n} q_i$$

for all $P = (p_1, \cdots, p_n), Q = (q_1, \cdots, q_n)\mathbb{R}_+^n$. Then f is a solution of the functional equation (1.6).

Proof. Define a function $\varepsilon : \mathbb{R}_+^n \times \mathbb{R}_+^n \to \mathbb{R}_+$ by $\varepsilon(P,Q) := \prod_{i=1}^n p_i \prod_{i=1}^n q_i$ for any $P = (p_1, \cdots, p_n), Q = (q_1, \cdots, q_n) \in \mathbb{R}_+^n$. Choose a sequence $W_k = (\frac{1}{k}, \cdots, \frac{1}{k})$ in $\mathbb{R}_+^n$. Then

$$\varepsilon(W_k \cdot P, Q) = \frac{1}{k^n}\prod_{i=1}^{n} p_i \prod_{i=1}^{n} q_i \to 0$$

as $k \to \infty$ for any $P = (p_1, \cdots, p_n), Q = (q_1, \cdots, q_n) \in \mathbb{R}_+^n$. By Theorem 4, f is a solution of the functional equation (1.6). □

Example 2.9. *Assume that a function $f : \mathbb{R}_+^3 \to X$ satisfy the inequality*

$$\begin{aligned}&\Big|\Big|f(p_1q_1, p_2q_2, p_3q_3) + f(p_1q_2, p_2q_3, p_3q_1) + f(p_1q_3, p_2q_1, p_3q_2)\\ &\quad - 3f(p_1, p_2, p_3) - 3f(q_1, q_2, q_3)\Big|\Big|\\ &\quad \leq p_1p_2p_3g_1q_2q_3\end{aligned}$$

for all $P = (p_1, p_2, p_3), Q = (q_1, q_2, q_3) \in \mathbb{R}_+^3$. Then we have

$$\begin{aligned}&f(p_1q_1, p_2q_2, p_3q_3) + f(p_1q_2, p_2q_3, p_3q_1) + f(p_1q_3, p_2q_1, p_3q_2)\\ &= 3f(p_1, p_2, p_3) + 3f(q_1, q_2, q_3)\end{aligned}$$

for all $P = (p_1, p_2, p_3), Q = (q_1, q_2, q_3) \in \mathbb{R}_+^3$.

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